Baryogenesis via Leptogenesis in an inhomogeneous universe

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We investigate the influence of primordial perturbations of the energy density and space–time metric on the generation of the lepton and baryon asymmetries. In the weak and strong washout regimes baryon isocurvature perturbations with amplitudes of the same order as those of the CMB perturbations are generated on scales of order of the respective Hubble scale. They are, however, completely washed out by baryon and photon diffusion at the later stages of the universe's evolution.

I. INTRODUCTION.

Recent oscillation experiments [1, 2] have confirmed that the neutrinos have small but nonvanishing masses. An observation of neutrinoless double beta decay by the future experiments [3] would hint toward the existence of very heavy right–handed Majorana neutrinos which generate naturally small neutrino masses via the see–saw mechanism. The existence of the heavy neutrinos can also explain [4] the observed baryon asymmetry of the universe since the three Sakharov conditions [5] are easily fulfilled in the standard model supplemented by the right–handed Majorana neutrinos.

The see–saw and baryogenesis via leptogenesis mechanisms are very attractive from a theoretical point of view. Unfortunately, there is still no compelling evidence for the existence of the heavy right–handed Majorana neutrinos. Apart from the neutrinoless double beta decay experiments one can search for its signatures in accelerator experiments or in lepton flavor violating decays [6, 7]. However the corresponding amplitudes are highly suppressed by the large masses of the right–handed neutrinos.

During leptogenesis, processes with right–handed neutrinos in initial, intermediate or final states are not suppressed. In a completely homogeneous and isotropic universe the outcome of baryogenesis via leptogenesis is a single number — the baryon–to–photon ratio, which is constant over space. In an inhomogeneous universe the primordial perturbations of the energy density and space–time metric modulate leptogenesis and the generated baryon–to–photon ratio depends on the spatial coordinates. Thus instead of a single number we get a *function* of space coordinates which can, in principle, provide us with some additional information on the model parameters.

A general perturbation with arbitrary fractional overdensity in each component can be represented as a sum of adiabatic and isocurvature perturbations. For a purely isocurvature perturbation the sum of the fractional overdensities is zero. For a purely adiabatic perturbation the fractional overdensity in each matter component (baryons, photons, etc.) is the same and hence the baryon–to–photon ratio is constant over space. Consequently, a deviation of the baryon–to–photon ratio from its average value amounts to a baryon isocurvature perturbation. We estimate here amplitude and scale dependence of the generated perturbations.

Baryon isocurvature perturbations are known to affect primordial nucleosynthesis. The abundances of the produced elements sensitively depend on the local baryon–to–photon ratio and the final average abundances can be considerably different from those in the homogeneous universe [8, 9, 10, 11]. They also affect the spectrum of the cosmic microwave background (CMB) radiation. Purely isocurvature perturbations produce a series of peaks in the CMB spectrum whose angular scales are approximately in the ratio $1 : 3 : 5\ldots$, whereas purely adiabatic density perturbations produce peaks whose angular scales are in the ratio $1 : 2 : 3\ldots$ [12]. Current large–scale observations are consistent with the primordial perturbations being adiabatic. It is however possible that the upcoming experiments will find a small admixture of the isocurvature perturbations at scales smaller than 10 Mpc [13, 14]; in particular the Planck satellite will be able to limit the amplitudes of isocurvature modes to less than 10% of the adiabatic mode [15, 16]. The isocurvature component modifies the sound velocity in the plasma and the initial conditions for the CMB [17], and is expected to "smear out" the spectrum, i.e. change height and width of the peaks.

The outline of the paper is as follows. In section II we derive differential and integral forms of the Boltzmann equation and prove the gauge invariance of the latter one. The derived integral Boltzmann equation is used to analyze leptogenesis in section III, where the generated lepton asymmetry is related to the adiabatic energy

density and metric perturbations and possible velocity perturbation in the heavy neutrino component. Finally we summarize our results in section IV.

II. THE BOLTZMANN EQUATION

The general form of the Boltzmann equation is [18]

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \hat{C}[f] \tag{1}$$

where λ is the affine parameter along a geodesic and $\hat{C}$ the collision operator acting on the one–particle distribution function f.

Since the Boltzmann equation treats particles as on–shell states, f is a function of space–time coordinates x^α and three independent components of the particle momentum P_i, or, alternatively, the particle kinetic energy E and unit momentum vector $\hat{p}_i$. Thus we can rewrite the left–hand side of (1) using the chain rule

$$\frac{\partial f}{\partial x^0} + \frac{\partial f}{\partial x^i}\frac{P^i}{P^0} + \frac{\partial f}{\partial E}\frac{1}{P^0}\frac{dE}{d\lambda} + \frac{\partial f}{\partial \hat{p}_i}\frac{1}{P^0}\frac{d\hat{p}_i}{d\lambda} = \frac{\hat{C}[f]}{P^0}. \tag{2}$$

The derivatives of E and $\hat{p}_i$ can be calculated using the geodesic equation; the result depends on the space–time metric.

The space–time metric of the early universe, which was almost homogeneous and isotropic, can be split into a background part

$$g_{\mu\nu}^0 = a^2(\eta)\mathrm{diag}(1,-1,-1,-1), \tag{3}$$

where η is the conformal time, and a small perturbation which reads [17]

$$\delta g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 2\phi & \mathcal{B}_{;i} \\ \mathcal{B}_{;i} & 2\left(\psi\delta_{ij} + \chi_{ij}\right) \end{pmatrix} \tag{4}$$

where a is the cosmic scale factor, ϕ, ψ and $\mathcal{B}$ are scalar functions, and χ_{ij} defined by

$$\chi_{ij} = \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\partial_k\partial_k\right)\mathcal{E} \tag{5}$$

is traceless. Note that the definition of ψ used here differs by the $\frac{1}{3}\partial_k\partial_k\mathcal{E}$ term from that used in [17]. This additional term will appear in the equation for the gauge transformation (13a) of ψ later on.

A nonzero shift function $\mathcal{B}$ means that comoving worldlines and worldlines orthogonal to hypersurfaces of constant time are not collinear and we are dealing with a locally nonorthogonal coordinate system [19]. We avoid the associated unnecessary complications by setting $\mathcal{B} = 0$ which is fulfilled, for instance, in all synchronous gauges and the longitudinal gauge.

Isotropy of the background space–time implies that the zero–order phase–space distribution functions are independent of the direction of the momentum vector, so

that $\frac{\partial f}{\partial \hat{p}_i}$ vanishes in the homogeneous universe defined by $g_{\mu\nu}^0$ and is of first order in the inhomogeneous universe defined by (4). Isotropy also implies that in the homogeneous universe the unit momentum vector is conserved, so that in the inhomogeneous universe $\frac{d\hat{p}_i}{d\lambda}$ is of first order as well. Consequently, the product of the two is of second order and the last term on the left–hand side of (2) can be neglected. Then the Boltzmann equation takes the form

$$\frac{\partial f}{\partial x^0} + \frac{\partial f}{\partial x^i}\frac{P^i}{P^0} - \frac{\partial f}{\partial E}\left(\frac{p^2}{E}\left[H - \dot{\psi} - \dot{\chi}_{ii}\,\hat{p}_i^2\right] + \phi_{,i}\,p\hat{p}_i\right)$$

$$= \sqrt{g_{00}}\,\frac{\hat{C}[f]}{E}$$

$$\sqrt{g_{00}} = a(1 + \phi). \tag{6}$$

The expression in the round brackets is (up to a sign) $\frac{1}{P^0}\frac{dE}{d\lambda}$ calculated to linear order in the small perturbations of the metric, and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. In the longitudinal gauge $\mathcal{B} = \chi_{ij} = 0$ and the Boltzmann equation (6) reverts to that derived in [18].

As we are only interested in the total generated asymmetry, we integrate the Boltzmann equation (6) over the phase space. Integration of the first term on the left–hand side gives the derivative of the particle number density n with respect to the time coordinate x^0.

Since for a massive particle $\frac{P^i}{P^0} = \frac{U^i}{U^0} = V^i$, the second term on the left–hand side can be rewritten in the form

$$\int \frac{\partial f}{\partial x^i}\frac{P^i}{P^0}d\Omega_p = \frac{\partial}{\partial x^i}\int fV^i d\Omega_p = \frac{\partial(nv^i)}{\partial x^i} \tag{7}$$

where v^i denotes the macroscopic three–velocity of the gas. In the early universe the macroscopic three–velocity is a first–order quantity and therefore (up to second order corrections) is proportional to the corresponding macroscopic four–velocity: $v^i \approx au^i$ (recall that $u^0 \approx a^{-1}$).

Since χ_{ij} is of first order, one can integrate the fifth term in (6) over the isotropic zero–order phase–space distribution function. In this approximation the integration gives the same result for any i and therefore (recall that χ_{ij} is traceless) this term vanishes to first order.

Analogously, since ϕ is of first order, one can again integrate the last term on the left–hand side of (6) over the phase–space using the zero–order phase–space distribution function; the integral vanishes due to the isotropy of the distribution function.

Upon integration by parts, the remaining terms take the form

$$3(H - \dot{\psi})n = \frac{n}{\sqrt{-g_3}}\frac{\partial\sqrt{-g_3}}{\partial x^0}, \quad \sqrt{-g_3} = a^3(1 - 3\psi). \tag{8}$$

Using the fact that χ_{ij} is traceless it is straightforward to check that g_3 is the determinant of the spatial part of the metric.

Collecting all the terms we obtain the integral form of the Boltzmann equation

$$\frac{1}{\sqrt{-g_3}}\frac{\partial(n\sqrt{-g_3})}{\partial x^0} + a\frac{\partial(nu^i)}{\partial x^i} = \sqrt{g_{00}}\int\frac{\hat{C}[f]}{E}d\Omega_p. \tag{9}$$

Equation (9) is a partial differential equation which is rather difficult to solve. However, if we neglect some further second–order terms, it can be substantially simplified. Since u^i as well as the gradient of n are of first order, their product can be neglected, so that $(nu^i)_{,i} \simeq nu^i_{,i}$. Next we introduce

$$\Upsilon \equiv n\sigma, \quad \sigma \equiv \sqrt{-g_3}\, \exp\left(\int u^i_{,i}\, a\, dx^0\right). \quad (10)$$

As can be checked by direct substitution, equation (9) is equivalent (up to the second order terms) to the following equation for Υ

$$\frac{1}{\sqrt{g_{00}}}\frac{\partial \Upsilon}{\partial x^0} = \sigma \int \frac{\hat{C}[f]}{E} d\Omega_p \quad (11)$$

where all the terms are evaluated at the *same* point in space.

Equation (11) is an *ordinary* differential equation with respect to the time coordinate. Any dependence of the generated lepton asymmetry on the space coordinates is "hidden" in the dependence of the metric and macroscopic parameters on the space coordinates. This substantial simplification is possible because we limit our analysis to linear perturbations here.

From (11) it is clear that Υ is conserved if the collision terms vanish; that is, in the inhomogeneous universe Υ replaces the particle number density per comoving volume Y.

Let us also note that equations (9) and (11) describe the development of the particle number density of a particular species and can be used in a universe with an arbitrary mixture of adiabatic and isocurvature perturbations.

The solutions of the Einstein equations give macroscopic parameters like the temperature as functions of the time coordinate. Since the collision terms depend explicitly on the inverse temperature $z = \frac{M_1}{T}$ (where M_1 is mass of the lightest Majorana neutrino) [20] rather than on the time coordinate x^0, it is convenient to rewrite (11) as follows:

$$\Upsilon' = \left(\frac{\sqrt{g_{00}}}{\dot{z}}\right)\sigma \int \frac{\hat{C}[f]}{E} d\Omega_p. \quad (12)$$

The dot denotes differentiation with respect to the time coordinate, whereas the prime denotes differentiation with respect to z.

The analysis of processes in the early universe is often complicated by the gauge choice. It is therefore important to check if the derived integral form of the Boltzmann equation is invariant with respect to the gauge transformations which preserve $\mathcal{B} = 0$. Let us consider a transformation from synchronous to longitudinal coordinates. Small perturbations of the metric and fluid velocity in the longitudinal gauge are related to those in the synchronous gauge by [17]

$$\psi_{(l)} = \psi_{(s)} - \frac{\dot{a}}{a}\dot{\mathcal{E}}_{(s)} - \frac{1}{3}\partial_k\partial_k\mathcal{E}_{(s)}, \quad (13a)$$

$$u_{i(l)} = u_{i(s)} + a\,\partial_i\dot{\mathcal{E}}_{(s)}. \quad (13b)$$

The same value of the proper time t (which is gauge invariant by definition) corresponds in the two gauges to two different values of the time coordinate: $x^0_{(s)}$ in the synchronous gauge and $x^0_{(l)}$ in the longitudinal one. Using the transformations laws for $\mathcal{B}$ and $\mathcal{E}$

$$\mathcal{B}_{(l)} = \mathcal{B}_{(s)} + \dot{\zeta} + x^0_{(s)} - x^0_{(l)}, \quad \mathcal{E}_{(l)} = \mathcal{E}_{(s)} + \zeta \quad (14)$$

(see [17] for more details) and taking into account that $\mathcal{B}$ is zero in both gauges and $\mathcal{E}$ is zero in the longitudinal gauge, we obtain

$$\Delta x^0 = x^0_{(s)} - x^0_{(l)} = \dot{\mathcal{E}}_{(s)}. \quad (15)$$

Using equations (8), (13a) and (15) we find the following transformation law for the determinant of the spatial part of the metric

$$\sqrt{-g_3}_{(l)}(x^0_{(l)}) = \sqrt{-g_3}_{(s)}(x^0_{(s)})\left(1 + \partial_k\partial_k\mathcal{E}_{(s)}\right). \quad (16)$$

Taking the divergence of (13b) and multiplying by the scale factor we obtain

$$au^i_{,i(l)}(x^0_{(l)}) = au^i_{,i(s)}(x^0_{(s)}) - \partial_k\partial_k\dot{\mathcal{E}}_{(s)}. \quad (17)$$

After integration over the time coordinate in (10) and asymptotic expansion of the exponent, the second term on the right–hand side of (17) cancels the analogous term in (16), so that $\sigma_{(l)}(x^0_{(l)}) = \sigma_{(s)}(x^0_{(s)})$. In other words, σ is invariant under the gauge transformation. Three–dimensional scalars like the particle number density are gauge invariant as well: $n_{(l)}(x^0_{(l)}) = n_{(s)}(x^0_{(s)})$ [21]. Together with the gauge invariance of σ this implies the gauge invariance of Υ.

Another important gauge invariant three–dimensional scalar is the energy density ρ

$$\rho = T^0_0 = \int f\sqrt{-g}\, P^0 dP^1 dP^2 dP^3 = \int fE d\Omega_p. \quad (18)$$

As can be shown by a direct calculation, for a gas of massless particles distributed according to the Boltzmann distribution the leading contribution to the energy density due to a small macroscopic velocity $\vec{u}$ is proportional to $\vec{u}^{\,2}$, i.e. is of second order and can be neglected. Thus $\rho = const. \cdot T^4$, and the temperature T (as well as the inverse temperature z) is gauge–invariant.

Therefore the left–hand side of (12) is gauge invariant since the derivative of one invariant quantity with respect to another invariant quantity is also invariant.

On the right–hand side of (12) the integral of the collision terms over the phase space is also invariant under

the gauge transformation [1]. Finally, since

$$\frac{\sqrt{g_{00}}}{\dot{z}} \equiv \frac{\sqrt{g_{00}}}{\left(\frac{\partial z}{\partial x^0}\right)} = \left(\frac{\partial z}{\partial t}\right)^{-1} \quad (19)$$

and the (inverse) temperature z as well as the proper time t are gauge invariant, the ratio (19) is also invariant. This completes our proof of the invariance of the integral Boltzmann equation under the gauge transformation (13).

Instead of the transformation (13) between synchronous and longitudinal gauge, we could have considered a transformation between two different gauges which fulfill $\mathcal{B} = 0$. The calculation would differ in only $\mathcal{E}$ being now different from zero in both gauges and appearing also on the left–hand side of equations (16) and (17), which however does not change the conclusion concerning the invariance of σ. Therefore, we may conclude that equation (12) is invariant not only under the transformation between longitudinal and synchronous gauge, but under all gauge transformations that preserve the gauge condition $\mathcal{B} = 0$.

III. INHOMOGENEOUS LEPTOGENESIS

To calculate the lepton asymmetry generated in the decays of the heavy Majorana neutrinos one has to solve a coupled system of Boltzmann equations for the heavy neutrino and lepton number densities.

To be able to perform this calculation analytically we neglect the contribution of the heavy neutrinos to the energy density of the universe. This is justified by the smallness of the associated number of effective massless degrees of freedom as compared to that of all the SM species. The two–body scattering processes and the flavor effects are also neglected for simplicity. In the simplest scenario of inflation isocurvature perturbations in the gas of the SM species are suppressed and we also neglect them. In particular, this implies that σ is the same for leptons and photons so that $\eta_L \equiv n_L/n_\gamma = \Upsilon_L/\Upsilon_\gamma$. The possible deviation of the macroscopic velocity of the heavy neutrino gas from that of leptons (i.e. velocity perturbation in the right–handed neutrino component) is parameterized by $r_u \equiv \frac{\sigma_\ell}{\sigma_\Psi} - 1 \approx \int (u_\ell^i - u_\Psi^i)_{,i} \, a \, dx^0$.

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[1] Perturbations of the space time metric may in principle lead to a change of decay widths and cross sections of the scattering processes. These effects, however, are not considered here. Following common practice we assume that the widths and cross sections are given by the same expressions as those in vacuum. Then the perturbations only affect the thermally averaged decay widths and scattering cross sections through the modification of the temperature and the related modification of the equilibrium phase–space distribution functions.

A. Approximate analytical solution of the system of Boltzmann equations

The collision terms are determined by the coupling of the Majorana neutrino to leptons and the Higgs

$$\mathcal{L} = N(\bar{\ell}\hat{\lambda}^\dagger \tilde{h}) - \tfrac{1}{2}\bar{N}\hat{M}N^c + \text{h.c.} \quad (20)$$

where ℓ denotes the leptons, $\tilde{h} = i\sigma_2 h^\dagger$ is the charge conjugate Higgs doublet, and N are the components of physical Majorana neutrino field $\Psi = \frac{1}{\sqrt{2}}(N + N^c)$.

Under the usual assumption that the phase–space distribution function of the Majorana neutrino is proportional to the equilibrium one, the contribution of the decay ($\Psi \to \ell h^\dagger$) and the inverse decay ($\ell h^\dagger \to \Psi$) processes to the lepton asymmetry reads [20]

$$\int \frac{\hat{C}[f_L]}{E} d\Omega_p = \langle \Gamma_{\Psi_1} \rangle \left[\varepsilon (n_\Psi - n_\Psi^{\text{eq}}) - \frac{n_\Psi^{\text{eq}}}{n_\ell^{\text{eq}}} n_\ell \right] \quad (21)$$

where ε is the usual CP asymmetry parameter and $\langle \Gamma_{\Psi_1} \rangle$ is the thermally averaged heavy neutrino decay width. Integrating over the equilibrium distribution function and neglecting a second–order contribution due to the small macroscopic velocity [2] we find

$$\langle \Gamma_{\Psi_1} \rangle = \Gamma_{\Psi_1} \frac{K_1(z)}{K_2(z)}, \quad \Gamma_{\Psi_1} = \frac{(\lambda\lambda^\dagger)_{11}}{8\pi} M_1, \quad (22)$$

where K_1 and K_2 are the modified Bessel functions. The equilibrium number densities are obtained by integration of the corresponding Maxwell–Boltzmann distribution functions over the phase space

$$n_\Psi^{\text{eq}} = \frac{T^3}{\pi^2} z^2 K_2(z), \quad n_\ell^{\text{eq}} = 4N\frac{T^3}{\pi^2} \quad (23)$$

where $N = 3$ is the number of generations. One should keep in mind that T refers to leptons and the Higgs; the right–handed neutrino is out of equilibrium and does not have a definite temperature.

The same processes also contribute to the right–hand side of the Boltzmann equation for the Majorana number density [20]

$$\int \frac{\hat{C}[f_\Psi]}{E} d\Omega_p = -\langle \Gamma_{\Psi_1} \rangle (n_\Psi - n_\Psi^{\text{eq}}). \quad (24)$$

In the homogeneous and isotropic universe dominated by radiation (see [17] for more details)

$$\sigma = \sqrt{-g_3} = a^3 \propto T_0^{-3}, \quad \left(\frac{\sqrt{g_{00}}}{\dot{z}}\right) = \frac{z_0}{\mathcal{H}} \quad (25)$$

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[2] In the gas rest frame the thermally averaged decay width depends only on temperature. After a Lorentz boost to the initial reference frame the decay width acquires the Lorentz factor, which differs from unity only by a second order term.

where (z_0) T_0 is the (inverse) background temperature and $\mathcal{H} \equiv H(M_1)$ is value of the Hubble parameter at a temperature equal to the mass of the lightest Majorana neutrino. In the inhomogeneous universe we tentatively write

$$\sigma \propto T^{-3}(1 + r_\sigma), \quad \left(\frac{\sqrt{g_{00}}}{\dot{z}}\right) = \frac{z}{\mathcal{H}}(1 + r_{\dot{z}}) \qquad (26)$$

where r_σ and $r_{\dot{z}}$ are first–order gauge–invariant[3] functions. In what follows we will use r_{σ_Ψ} and r_σ for the heavy neutrino and SM species, respectively. Introducing finally $\kappa \equiv \Gamma_{\Psi_1}/\mathcal{H}$, $\Delta \equiv \Upsilon_\Psi - \Upsilon_\Psi^{eq}$ and

$$\gamma_D(z) \equiv \frac{K_1(z)}{K_2(z)}, \quad \gamma_L(z) \equiv \frac{z^2}{4N}K_2(z) \qquad (27)$$

we obtain the following system of Boltzmann equations for the lepton asymmetry and Majorana neutrino number density

$$\Delta' = -\kappa z\,(1 + r_{\dot{z}})\,\gamma_D\Delta - \Upsilon_\Psi'^{eq} \qquad (28a)$$
$$\Upsilon_L' = \kappa z\,(1 + r_{\dot{z}})\,[\varepsilon(1 + r_u)\gamma_D\Delta - \gamma_L\Upsilon_L] \qquad (28b)$$

where the relation $r_u = r_\sigma - r_{\sigma_\Psi}$ has been used. If r_σ, r_{σ_Ψ} and $r_{\dot{z}}$ vanish, the resulting system coincides with that in the homogeneous universe and the generated asymmetry depends only on ε and κ.

In the weak washout regime ($\kappa \ll 1$) the deviation from equilibrium is large and washout processes play almost no role, so that $\Upsilon_L' \approx -\varepsilon(1 + r_u)\Upsilon_\Psi'$. Together with the initial conditions $\Delta(0) = \Upsilon_L(0) = 0$ this gives upon integration by parts and use of the zero–order solution of (28a), $\Upsilon_\Psi^0(z) = \Upsilon_\Psi(0)\exp(-\kappa z^2/2)$, (see [22])

$$\Upsilon_L \approx \Upsilon_L^0\left[1 + r_\sigma(0) + \int r_u'\exp\left(-\kappa z^2/2\right)dz\right], \qquad (29)$$

where the unperturbed value of the generated asymmetry in the weak washout regime is given by

$$\Upsilon_L^0 = \frac{2\varepsilon}{\pi^2}. \qquad (30)$$

In the strong washout regime ($\kappa \gg 1$) the deviation from equilibrium is small, so that the final lepton asymmetry is independent of the initial conditions and determined by the dynamics of leptogenesis. In this regime the density of Majorana neutrinos closely tracks its equilibrium value, which implies $\Delta' \simeq 0$ [22]. Substitution of (28a) into (28b) then gives

$$\Upsilon_L' = -\varepsilon(1 + r_u)\Upsilon_\psi'^{eq} - \kappa z\,(1 + r_{\dot{z}})\gamma_L\Upsilon_L. \qquad (31)$$

The formal solution of this equation satisfying the initial conditions $\Delta(0) = \Upsilon_L(0) = 0$ reads

$$\Upsilon_L = -\varepsilon\int_0^z (1 + r_u)\Upsilon_\Psi'^{eq}\exp\left(-\int_{z'}^z \kappa z''(1 + r_{\dot{z}})\gamma_L dz''\right)dz'. \qquad (32)$$

For $z \gg 1$ one can use the large–argument asymptotics of the Bessel functions K_1 and K_2, so that approximately

$$-\Upsilon_\psi'^{eq}(z) \approx [1 + r_{\sigma_\Psi}(z) - r_{\sigma_\Psi}'(z)]\sqrt{\frac{z^3}{2\pi^3}}\exp(-z). \qquad (33)$$

The arguments of the exponents in (32) and (33) combine to a function which has a sharp peak at $z = z_f$, where z_f is the solution of

$$\kappa z_f\gamma_L(z_f)[1 + r_{\dot{z}}(z_f)] = 1. \qquad (34)$$

Consequently one can apply the method of steepest descent to estimate the asymptotic value of this integral. Approximating γ_L by its large–argument asymptotics and keeping only the terms of highest order in z_f we obtain

$$-(\kappa z\gamma_L[1 + r_{\dot{z}}])'_{z=z_f} \approx 1 - r_{\dot{z}}'(z_f), \qquad (35a)$$
$$\int_{z_f}^\infty \kappa z\gamma_L[1 + r_{\dot{z}}]\,dz \approx 1 + r_{\dot{z}}'(z_f). \qquad (35b)$$

A deviation of $r_{\dot{z}}$ from zero leads to a deviation of z_f from its unperturbed value. Using (34) one can show that the shift is given by $z_f - z_{f_0} \approx r_{\dot{z}}(z_f)$, where z_{f_0} denotes value of the freeze–out temperature in the $r_{\dot{z}} = 0$ case.

Using equations (34) and (35) we get for the asymptotic value of the asymmetry [4]

$$\Upsilon_L \approx \Upsilon_L^0\frac{1 + r_\sigma(z_f) - r_\sigma'(z_f) + r_u'(z_f)}{1 + (1 + z_f^{-1})r_{\dot{z}}(z_f) + \frac{1}{2}r_{\dot{z}}'(z_f)} \qquad (36)$$

where the unperturbed value of the generated asymmetry in the strong washout regime is given by

$$\Upsilon_L^0 = \frac{2\varepsilon}{\pi^2}\frac{N}{\kappa z_{f_0}}\frac{2\sqrt{2\pi}}{\exp(1)}. \qquad (37)$$

Despite its simplicity, for $\kappa \gtrsim 10$ the analytical solution (36) deviates from the exact numerical solution of (28) by only a few percent and correctly describes the dependence of the asymmetry on r_σ and $r_{\dot{z}}$. For large κ the inverse freeze–out temperature $z_f \gg 1$ and the term proportional to z_f^{-1} in the denominator can be neglected.

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[3] The gauge invariance of r_σ and $r_{\dot{z}}$ follows from the gauge invariance of the (inverse) temperature and of σ and $\frac{\partial z}{\partial t}$.

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[4] On very small scales the thermodynamic quantities rapidly oscillate and the solution obtained by the method of steepest descent under the assumption that the functions under the integral are smooth and do not oscillate is not valid.

B. The baryon–to–photon ratio

To complete the analysis we have to evaluate r_σ and $r_{\dot{z}}$. Introducing the dimensionless temperature perturbation $\Theta \equiv \frac{\delta T}{T}$ one can relate z to its background value $z = z_0(1 + \Theta)^{-1}$. Using the known dependence of z_0 on the time–coordinate in the radiation–dominated universe, $z_0 = \mathcal{H} x^0 \propto a$, and the resulting relation $\dot{z}_0 z_0^{-1} = H$ we obtain for time derivative of z

$$\frac{dz}{dt} = \frac{1}{\sqrt{g_{00}}}\frac{\partial z}{\partial x^0} = \frac{\mathcal{H}}{z_0(1+\Theta)}\frac{1}{\sqrt{g_{00}}}\left(1 - \frac{1}{H}\frac{\partial\Theta}{\partial x^0}\right), \quad (38)$$

which implies

$$r_{\dot{z}} = \phi + 2\Theta + H^{-1}\dot{\Theta}. \quad (39)$$

Expanding the exponent in (10) and neglecting second–order terms one can write σ in the form

$$\sigma = a^3(x^0)\left(1 - 3\psi + \int a u^i_{,i}\, dx^0\right), \quad (40)$$

where the scale factor a is related to the background temperature by $a^3 \propto T_0^{-3} = T^{-3}(1 + 3\Theta)$. Consequently

$$r_\sigma = 3\Theta - 3\psi + \int a u^i_{,i} dx^0. \quad (41)$$

Since r_σ and $r_{\dot{z}}$ are gauge invariant, it is sufficient to evaluate them in one particular gauge. In the longitudinal gauge solution of the Einstein equations for the Fourier components of ϕ and Θ in the radiation–dominated universe reads [17]

$$\phi_k = C_1\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) + C_2\left(\frac{\cos x}{x^3} + \frac{\sin x}{x^2}\right) \quad (42a)$$

$$\Theta_k = \frac{C_1}{2}\left[\left(\frac{2-x^2}{x^2}\right)\left(\frac{\sin x}{x} - \cos x\right) - \frac{\sin x}{x}\right]$$
$$+ C_2\left[\left(\frac{1-x^2}{x^2}\right)\left(\frac{\cos x}{x} + \sin x\right) + \frac{\sin x}{2}\right] \quad (42b)$$

where $x \equiv \frac{kx^0}{\sqrt{3}}$. The initial conditions are specified by C_1 and C_2, which are functions of k. The divergence of the macroscopic fluid velocity $v \equiv u^i_{,i}$ can be calculated using the Einstein equation for the off–diagonal components of the Ricci tensor. In the absence of anisotropic stress, $\psi = \phi$, and these simplify to $(a\phi)_{,i} = 4\pi G a^2(\varepsilon_0 + p_0)u_i$ [17]. After use of the Friedmann equation, $3H^2 = 8\pi G a^2\varepsilon_0$, and some algebra we obtain for the corresponding Fourier components in the flat radiation–dominated universe

$$v_k = \frac{k^2}{2\dot{a}^2}(a\dot{\phi}_k). \quad (43)$$

Trading the integration with respect to the time coordinate for the integration with respect to x and taking into account the $\psi = \phi$ equality we find for the Fourier

transform of r_σ [5]

$$r_{\sigma_k} = 3\Theta_k - 3\phi_k + \int \frac{3x}{2}\frac{\partial}{\partial x}(x\phi_k)dx = 0. \quad (44)$$

Using the relation $\dot{\Theta} = \dot{x}\frac{\partial}{\partial x}\Theta = Hx\frac{\partial\Theta}{\partial x}$ we obtain a very simple expression for the Fourier transform of $r_{\dot{z}}$ [6]

$$r_{\dot{z}_k} = \phi_k + 2\Theta_k + x\frac{\partial\Theta_k}{\partial x} = -\frac{x}{2}\left(C_1\sin x + C_2\cos x\right). \quad (45)$$

The lepton to photon number ratio, $\eta_L = \Upsilon_L/\Upsilon_\gamma$, can now be easily read off from (29) and (36). The quantity of interest is the departure of the asymmetry from its background value, $r_\eta \equiv \eta_L/\eta_L^0 - 1$. Since the baryon asymmetry is linearly related to the lepton one, r_η also characterizes the baryon isocurvature perturbations. Neglecting the term proportional to z_f^{-1} in the denominator of (36), we find in the strong washout regime [7]

$$r_\eta = r'_u(z_f) - r_{\dot{z}}(z_f) - \tfrac{1}{2}r'_{\dot{z}}(z_f). \quad (46)$$

The calculation of the CMB anisotropies is performed in momentum space, so that one is interested in the Fourier components of (46). The Fourier transform must be evaluated at the inverse freeze–out temperature z_f; it corresponds to $x_f = z_f\mathcal{X}$, where $\mathcal{X} \equiv \frac{k\mathcal{R}}{\sqrt{3}}$ is the value x takes at $z = 1$, and $\mathcal{R} \equiv \mathcal{H}^{-1} \propto M_{Pl}M_1^{-2}$ is the Hubble scale at this stage. The derivative of (45) with respect to the inverse temperature can be obtained upon use of the zero–order relation $\frac{\partial}{\partial z} = \mathcal{X}\frac{\partial}{\partial x}$. Using the introduced notation we find for the Fourier transform of r_η in the strong washout regime

$$r_{\eta_k} = \mathcal{X}\left[\frac{\partial}{\partial x}r_u + \tfrac{1}{4}(2C_1\, z_f - C_2)\sin\mathcal{X}z_f + \tfrac{1}{4}(2C_2 z_f + C_1)\cos\mathcal{X}z_f\right]. \quad (47)$$

In this regime the heavy neutrinos are tightly coupled to the SM species and it is natural to expect that r_u vanishes. The typical size of the perturbations is given by the Hubble radius $\mathcal{R}$. On larger scales r_{η_k} is suppressed

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[5] Equation (44) implies that for the photons $\Upsilon_\gamma = \Upsilon_\gamma^0 = \frac{2}{\pi^2}$ is constant in the early universe, just as expected. Since photons are in equilibrium in the early universe, the right–hand side of the Boltzmann equation for photons vanishes and Υ_γ is conserved.

[6] A calculation using the solution of the Einstein equations for small perturbations in the synchronous gauge given in [23] leads to the same expressions for r_σ and $r_{\dot{z}}$; this confirms the gauge invariance of the introduced functions.

[7] Since the contribution of the heavy neutrinos to the energy density is neglected, the baryon–to–photon ratio is related to the baryon–to–entropy ratio by a constant factor at this stage of the universe's evolution. Later, as the SM species decouple as they become massive, the baryon–to–entropy ratio remains constant, wheres the baryon–to–photon ratio changes. This change is absorbed into the definition of the background value of the asymmetry and does not modify the expressions for r_η.

by the overall factor $\mathcal{X}$, which becomes small. On superhorizon scales $r_{\eta_k} \approx \frac{1}{4}\mathcal{X}(2C_2 z_f + C_1)$ and vanishes as $\kappa \to \infty$.

From (29) it follows that in the weak washout regime

$$r_\eta = \int \left(\frac{\partial}{\partial x} r_u \right) \exp\left(-\kappa x^2 / 2\mathcal{X}^2 \right) dx \,, \qquad (48)$$

where we have used the relation $x = z\mathcal{X}$. That is, a nonzero r_η is generated in this regime only if the velocity of the right–handed neutrino deviates from that of the SM species. The typical scale and amplitude of r_u are model–dependent. Nevertheless one can make a plausible guess about these quantities: if perturbations in the SM and heavy neutrino components are created by the same mechanism, then the typical amplitude and scale of r_u cannot exceed those of the velocity perturbation of the SM species. Since we are interested in large–scale perturbations, which correspond to small $\mathcal{X}$, it is sufficient to know the behaviour of r_u in the vicinity of $x = 0$

$$r_u = \sum a_n x^n \,, \quad a_n \sim \mathcal{O}(C_{1,2}) \,. \qquad (49)$$

If, for example, u_ψ vanishes in the longitudinal coordinates[8], then as follows from (43) $a_{-1} = 3C_2$, $a_0 = \frac{1}{2}C_1$, $a_2 = \frac{1}{4}C_1 \ldots$, whereas if u_ψ vanishes in synchronous coordinates, then $a_1 = \frac{3}{2}C_2$, $a_4 = \frac{1}{32}C_1 \ldots$ (see Eq. (15) in [23]). To integrate the terms with negative n we cut the integral at some x_{min} such that $a_n x_{min}^n \ll 1$ to insure the applicability of the perturbative treatment. We furthermore consider only very large scales $\mathcal{X} \ll x_{min}$. Then

$$r_{\eta_k} = a_1 \sqrt{\pi/2\kappa}\, \mathcal{X} + \mathcal{O}(\mathcal{X}^2) \,. \qquad (50)$$

That is, in the weak washout regime r_u is also suppressed on large scales.

C. BBN and CMB

In the radiation–dominated universe the horizon size grows much faster than the scale factor, and consequently the generated perturbations are not protected from smoothing by baryon diffusion at later stages of the universe's evolution.

Let us first consider the strong washout regime. Relation (47) holds for arbitrary mass of the Majorana neutrino. For the right–handed neutrino mass $M_1 \sim 10^9$ GeV, scales of the order of $\mathcal{R}$ at the time of leptogenesis correspond to (roughly speaking) 10^{-4} m at neutrino decoupling. This is much smaller than the baryon diffusion length ~ 200 m [25] at this stage. Therefore in this case

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[8] The analysis performed in [24] in longitudinal gauge demonstrates that velocity perturbation of the heavy dark matter particles are small compared to those of the massless species.

the perturbations of the baryon to photon ratio are completely smoothed out before nucleosynthesis starts. In the electroweak–scale resonant leptogenesis scenario [26] the Hubble scale at the time of leptogenesis corresponds to (roughly speaking) ~ 10 km at 0.1 MeV. At the latter temperature, which corresponds to the beginning of nucleosynthesis, the neutron diffusion length ~ 200 km. In other words, the baryon perturbations are not smoothed out on scales which correspond to $\mathcal{X} \lesssim 0.1$ at the time of leptogenesis. Given the typical size of the integration constants $C_{1,2} \sim 10^{-5} - 10^{-4}$ the amplitude of the perturbations not washed out by baryon diffusion is too small to affect nucleosynthesis in an observable way.

After the end of nucleosynthesis and before the beginning of recombination there are no neutral baryons and the diffusion length substantially decreases. Thus the baryon isocurvature perturbations which have not been smoothed out survive till recombination – the time of CMB formation. At this stage small scale perturbations blueof the photon gas undergo a mechanism called *diffusion damping* or *Silk damping* [27, 28]. The photons diffuse from regions with higher density to lower density regions and thereby effectively smooth out baryon inhomogeneities due to their large mean free path. This damps the amplitudes of acoustic oscillations by a factor

$$D \equiv \exp\left(\int_{t_A}^{t_0} \Gamma \mathrm{d}x^0 \right) \equiv \exp\left[-\left(\frac{M_c}{M} \right)^{2/3} \right], \qquad (51)$$

where Γ is the damping rate, t^0 the present time and M_c is called the critical mass that can only be obtained approximately from the integral over the damping rate. For a radiation-dominated universe during the phase of acoustic oscillations, one obtains [29]

$$M_c \approx 8 \times 10^{13} (\Omega_0 h_0^2)^{-1/2} M_\odot \qquad (52)$$

with density parameter Ω_0 and rescaled Hubble parameter h_0 evaluated at the present time and $M_\odot$ denoting the solar mass. Sufficient damping is then obtained for (M_c/M) of the order of 10 or larger. Translated into length scales, this means that all scales

$$\lambda_D \lesssim 3 \times 10^{18} \mathrm{m} \qquad (53)$$

are affected by Silk damping at $T \sim 10$ eV, which roughly corresponds to the temperature at recombination, and only perturbations on scales larger than that can survive. For the right–handed neutrino mass $M_1 \sim 10^9$ GeV, scales of the order of $\mathcal{R}$ at the time of leptogenesis correspond to scales of the order of 10 m at recombination. This is obviously smaller than the damping scale λ_D by so many orders of magnitude that no observable imprint on the CMB can be produced. Turning the argumentation around, we can calculate the mass scale M_1 that would be necessary to obtain perturbations on large enough scales to prevent them from being smoothed out by Silk damping, taking into account that $\mathcal{R}$ scales as T^{-2} in a radiation–dominated universe. The

result is very disappointing: one would need a mass scale of $M_1 \sim 100$ eV which is far below the scale where the sphalerons are in thermal equilibrium and can convert a lepton asymmetry into a baryon asymmetry. Therefore we conclude that in the strong-washout regime, the baryon inhomogeneities coming from the spatial fluctuation of the efficiency of leptogenesis can neither affect BBN nor the CMB in an observable way.

Given the estimate (50), the conclusions drawn in the case of the strong washout regime also apply to the case of the weak washout regime.

IV.   SUMMARY AND CONCLUSIONS

In this short paper we have investigated the influence of the primordial perturbations of the energy density and metric on the efficiency of thermal leptogenesis.

To perform the analysis we have derived an integral form of the Boltzmann equation for a gauge–invariant combination Υ of particle number density, the macroscopic fluid velocity and space–time metric. In the absence of the collision terms Υ is conserved to linear approximation and replaces the particle number density per comoving volume in the inhomogeneous universe.

Using the integral form of the Boltzmann equation approximate analytical solutions for the lepton asymmetry in the weak and strong washout regimes have been found.

In the strong washout regime the generated baryon isocurvature perturbations are correlated with the perturbations of the energy density and metric. In the weak washout regime the solution deviates from the background one only in the presence of the heavy neutrino velocity perturbation r_u. The typical scale of the perturbations at the time of leptogenesis is set by the Hubble scale $\mathcal{R}$ in both regimes. The amplitude is inversely proportional to the scale.

The generated perturbations are smoothed by baryon diffusion at the later stages of the universe's evolution. The scale of the generated perturbations is comparable to the neutron diffusion length at the beginning of BBN only if the right–handed neutrino is as light as in the electroweak–scale resonant leptogenesis scenario. However even in this extreme case the generated perturbations are smoothed out by the *Silk damping* at the recombination. For this reason the influence of the baryon inhomogeneities coming from the spatial fluctuation of the efficiency of leptogenesis on BBN and CMB is unobservably small.

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